The properties of Pascal’s triangle

**ACARA**

* derive and use simple identities associated with [Pascal’s triangle](http://www.australiancurriculum.edu.au/Glossary?a=SSCMSM&t=Pascal%E2%80%99s%20triangle). (ACMSM009)

Pascal’s triangle

Pascal’s triangle is an arrangement of numbers. In general the *n*th row consists of the binomial Coefficients

*nCr* or with the *r* = 0, 1,…, *n*

1



In Pascal’s triangle any term is the sum of the two terms ‘above’ it.

For example 10 = 4 + 6.

Identities include:

The recurrence relation,.



<http://www.australiancurriculum.edu.au/Glossary?a=SSCMSM&t=Pascal%E2%80%99s%20triangle>

Prove nCk = n-1Ck-1 + n-1Ck





Prove 





**Some other basic properties**:

Pascal’s triangle can be used to find the coefficients of the binomial expansion (a + b)n.

For example (a + b)6 = a6 + 6a5b1 +15a4b2 +20a3b3 +15a2b4 +6a1b5 +a6

where the coefficients are the elements of the 7th row of Pascal’s triangle 1 6 15 20 15 6 1

Given the 6 factors of (a + b)6 are (a + b)(a + b)(a + b)(a + b)(a + b)(a + b) then, for example, to get the 15a4b2 term, we have to select which of the terms contributes the four “a”s and the remaining terms contribute the “b”s. This can be done in 6C4 (or 6C2) ie 15 ways.

so we have

1 6 15 20 15 6 1 is the same as 6C0 6C1 6C2 6C3 6C4 6C5 6C6

Verify that this is true!

so (a + b)6 = 6C0a6 + 6C1a5b1 +6C2a4b2 +6C3a3b3 +6C4a2b4 +6C5a1b5 +6C6a6.

**What is the sum of the elements in a row in Pascal’s triangle?**

If we put a = b = 1 then we obtain

(1 + 1)6 = 6C0+ 6C1+6C2+ 6C3+6C4+ 6C5+ 6C6

so 26 = 6C0+ 6C1+6C2+ 6C3+6C4+ 6C5+ 6C6

In general

2n = nC0+  nC1+ nC2+  nC3+…… nCn-2 +  nCn-1+ nCn

i.e.  or 

Application: In how many ways can a selection be made from 10 different lollies?

 10C1 + 10C2 + 10C3 + ….. 10C9 + 10C10 = 210-  10C0

= 210 -1

Another property that follows is

6C0+ 6C2+6C4+ 6C6=6C1+6C3+6C5

i.e. **1** 6 **15** 20 **15** 6 **1**

**⇒1 + 15 + 15 + 1 =** 6 + 20 + 6

i.e. the sum of the “odd” terms is equal to the sum of the “even” terms.

Check that this works for any row in Pascal’s triangle.



There are many patterns in Pascal’s triangle that are well known.

Note: If in any row in Pascal’s triangle, the number after the one is prime, then all of the elements in that row with the exception of one, are multiples of the prime.

Explain using n and k.

The following diagram may be useful in the next few investigations:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  | 1 |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  | 1 |   | 1 |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  | 1 |   | 2 |   | 1 |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | 1 |   | 3 |   | 3 |   | 1 |  |  |  |  |  |  |  |
|  |  |  |  |  |  | 1 |  | 4 |   | 6 |   | 4 |   | 1 |  |  |  |  |  |  |
|  |  |  |  |  | 1 |   | 5 |   | 10 |   | 10 |   | 5 |   | 1 |  |  |  |  |  |
|  |  |  |  | 1 |   | 6 |   | 15 |   | 20 |   | 15 |   | 6 |   | 1 |  |  |  |  |
|  |  |  | 1 |  | 7 |   | 21 |   | 35 |   | 35 |   | 21 |   | 7 |   | 1 |  |  |  |
|  |  | 1 |   | 8 |   | 28 |   | 56 |   | 70 |   | 56 |   | 28 |   | 8 |   | 1 |  |  |
|  | 1 |   | 9 |   | 36 |   | 84 |   | 126 |   | 126 |   | 84 |   | 36 |   | 9 |   | 1 |  |
| 1 |   | 10 |   | 45 |   | 120 |   | 210 |   | 252 |   | 210 |   | 120 |   | 45 |   | 10 |   | 1 |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  | 0C0 |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  | 1C0 |   | 1C1 |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  | 2C0 |   | 2C1 |   | 2C2 |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | 3C0 |   | 3C1 |   | 3C2 |   | 3C3 |  |  |  |  |  |  |  |
|  |  |  |  |  |  | 4C0 |  | 4C1 |   | 4C2 |   | 4C3 |   | 4C4 |  |  |  |  |  |  |
|  |  |  |  |  | 5C0 |   | 5C1 |   | 5C2 |   | 5C3 |   | 5C4 |   | 5C5 |  |  |  |  |  |
|  |  |  |  | 6C0 |   | 6C1 |   | 6C2 |   | 6C3 |   | 6C4 |   | 6C5 |   | 6C6 |  |  |  |  |
|  |  |  | 7C0 |  | 7C1 |   | 7C2 |   | 7C3 |   | 7C4 |   | 7C5 |   | 7C6 |   | 7C7 |  |  |  |
|  |  | 8C0 |   | 8C1 |   | 8C2 |   | 8C3 |   | 8C4 |   | 8C5 |   | 8C6 |   | 8C7 |   | 8C8 |  |  |
|  | 9C0 |   | 9C1 |   | 9C2 |   | 9C3 |   | 9C4 |   | 9C5 |   | 9C6 |   | 9C7 |   | 9C8 |   | 9C9 |  |
| 10C0 |   | 10C1 |   | 10C2 |   | 10C3 |   | 10C4 |   | 10C5 |   | 10C6 |   | 10C7 |   | 10C8 |   | 10C9 |   | 10C10 |

1. Find an expression for Tn i.e. the nth triangular number in terms of nCk. Hence find T30.

Write down a much simpler algebraic expression for Tn and explain how it relates to the expression in terms of nCk.

2. Find a summation expression in terms of nCk for

(a) the sum of the set of counting numbers from 1 to 10.

(b) the sum of the set of triangular numbers from 1 to n.

3. Interpret nCk = n-1Ck-1 + n-1Ck referring to cells in Pascal’s triangle.

**ANSWERS**

1. Tn = n+1C2 T30 = 31C2 = 465

 Tn=  This term is the simplification of n+1C2.

2. (a)  (b) 

3. 8C3+8C4 = 9C4

Add two adjacent cells in Pascal’s triangle and you get the cell under and

betweenthem.

**SQUARE NUMBERS**

Consider the triangular numbers 1, 3, 6, 10, 15, 21 etc

Any two adjacent triangular numbers add to give a square number. eg 3 + 6 = 9, 15 + 21 = 36

By using the formula for triangular numbers in terms of nCk, find and simplify Tn + Tn+1  to prove that the sum of two adjacent triangular numbers is in fact a square number.

**ANSWER**

Tn + Tn+1  = n+1C2 + n+2C2



which is a square number.

**POWERS of 11**

|  |  |  |
| --- | --- | --- |
| Row number | Powers of 11 | Pascal's triangle |
| 0 | 110 = 1 | 1 |
| 1 | 111= 11 | 1 1 |
| 2 | 112=121 | 1 2 1 |
| 3 | 113= 1331 | 1 3 3 1 |
| 4 | 114=14641 | 1 4 6 4 1 |
| 5 | 115=161051 | 1 5 10 10 5 1 |
| 6 | 116=1771561 | 1 6 15 20 15 61 |
| 7 | 117 = | 1 7 21 3535 21 7 1 |
| 8 | 118 = | 1 8 28 56 70 56 28 8 1 |
|  |  |  |

Write down the relationship between the power of 11 and the numbers in the corresponding row of Pascal’s triangle.

Can you see how this works for 115 and 116?

Use this to determine the values of 117 and 118.

**STAR of DAVID**

inspired by <http://threesixty360.wordpress.com/2008/12/21/star-of-david-theorem/>



Draw the Star of David  on the Pascal’s triangle below using the  expressions in the diagram above. Note : You will have to rotate the vertices 60o to the right or left.

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Consider the triangles



Confirm that a×c×e = b×d ×f . Check with another Star of David on Pascal’s triangle.

The products of the numbers at the vertices of each triangle on the Star of David pattern are the same.

Prove this is true for all positions on Pascal’s triangle where the Star of David can be drawn.

**Hint: Let the point in the middle of the Star of David be represented by nCk.**

To make it easier:

Steps to consider:

Find the relevant corners of the star in terms of n and k



nCk

Further hint

 

n-1Cr-1

nCr-1

n+1Cr

n+1Cr+1

nCr+1

n-1Cr

nCk

Show = 

**ANSWER**

Show = 





as required.

**The Fibonacci sequence** is nested in Pascal’s triangle.



It is a lot easier to see if Pascal’s triangle is drawn differently as below and coloured appropriately.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 |  |  |  |  |  |  |  |  |  |  |  |  |
| 1 | 1 |  |  |  |  |  |  |  |  |  |  |  |
| 1 | 2 | 1 |  |  |  |  |  |  |  |  |  |  |
| 1 | 3 | 3 | 1 |  |  |  |  |  |  |  |  |  |
| 1 | 4 | 6 | 4 | 1 |  |  |  |  |  |  |  |  |
| 1 | 5 | 10 | 10 | 5 | 1 |  |  |  |  |  |  |  |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 |  |  |  |  |  |  |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |  |  |  |  |  |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |  |  |  |  |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |  |  |  |
| 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |  |  |
| 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 |  |
| 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 |
|  | Sum |  |  |  |  |  |  |  |  |  |  |  |
|   | 1 |  |  |  |  |  |  |  |  |  |  |  |
|   | 1 |  |  |  |  |  |  |  |  |  |  |  |
|   | 2 |  |  |  |  |  |  |  |  |  |  |  |
|   | 3 |  |  |  |  |  |  |  |  |  |  |  |
|   | 5 |  |  |  |  |  |  |  |  |  |  |  |
|   | 8 |  |  |  |  |  |  |  |  |  |  |  |
|   | 13 |  |  |  |  |  |  |  |  |  |  |  |
|   | 21 |  |  |  |  |  |  |  |  |  |  |  |
|   | 34 |  |  |  |  |  |  |  |  |  |  |  |
|   | 55 |  |  |  |  |  |  |  |  |  |  |  |
|   | 89 |  |  |  |  |  |  |  |  |  |  |  |
|   | 144 |  |  |  |  |  |  |  |  |  |  |  |
|   | 233 |  |  |  |  |  |  |  |  |  |  |  |

Find an expression for the nth term of the Fibonacci sequence in terms of nCk.

Hence find F15.

NB Recursive formulae depend on knowing the previous terms. This expression enables you to calculate any terms of the Fibonacci sequence knowing just the term number n, so although awkward, it has advantages!

**ANSWER**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 0C0 |  |  |  |  |  |  |  |
| 1C0 | 1C1 |  |  |  |  |  |  |
| 2C0 | 2C1 | 2C2 |  |  |  |  |  |
| 3C0 | 3C1 | 3C2 | 3C3 |  |  |  |  |
| 4C0 | 4C1 | 4C2 | 4C3 | 4C4 |  |  |  |
| 5C0 | 5C1 | 5C2 | 5C3 | 5C4 | 5C5 |  |  |
| 6C0 | 6C1 | 6C2 | 6C3 | 6C4 | 6C5 | 6C6 |  |
| 7C0 | 7C1 | 7C2 | 7C3 | 7C4 | 7C5 |  |  |
| 8C0 | 8C1 | 8C2 | 8C3 | 8C4 |  |  |  |
| 9C0 | 9C1 | 9C2 | 9C3 |  |  |  |  |
| 10C0 | 10C1 | 10C2 |  |  |  |  |  |
| 11C0 | 11C1 |  |  |  |  |  |  |
| 12C0 |  |  |  |  |  |  |  |

F1 = 1 = 0C0

F2 = 1 = 1C0

F3 = 2 = 2C0 + 1C1

F4 = 3 = 3C0 + 2C1

F5 = 5 = 4C0 + 3C1+ 2C2

F6 = 8 = 5C0 + 4C1+ 3C2

F7 = 13 = 6C0+ 5C1+ 4C2+3C3

F8 = 21 = 7C0+ 6C1+ 5C2+4C3

If n is evenFn = 

If n is oddFn = 

**ANSWER**

F15=?

n is odd so F15 = 

 = 1 + 13 + 66 + 165 + 210 +126+28+ 1

 = 610

**The Hockey Stick Pattern**



<http://britton.disted.camosun.bc.ca/pascal/pascal.html>

The hockey stick pattern is formed by starting at any edge, going down any number of cells then turning to form the foot of the hockey stick.

It has been conjectured that the sum of the stem of the hockey stick (including the heel) is equal to the number in the toe cell. Check using the examples in the diagram above.

Prove the conjecture using cells in terms such as nCk .

**Hint :nCk = n-1Ck-1 + n-1Ck**

**ANSWER**

1+7+28+84+210+462+924 = 1716

1+12 = 13

1+6+21+456 = 84 It works!

and now to prove it works for all cases!

The sum of the stem = nC0+ n+1C1+ n+2C2+ …+ r-1Ck

The toe = rCk

Remember nCk = n-1Ck-1 + n-1Ck

The three hockey sticks illustrated consist of the patterns:

(a) 6C0, 7C1, 8C2,9C3, 10C4, 11C5, and12C6 and the length of the toe is 13C6

 which is equivalent to

 6C6, 7C6, 8C6,9C6, 10C6, 11C6, and12C6 and the length of the toe is 13C7

(b) 11C11,and12C11 and the length of the toe is 13C12

(c) 5C5,6C5, 7C5 and8C5 and the length of the toe is 9C6.

Considering the pattern of the hockey sticks, and if the toe is defined to be nCr

then the stem elements are n-1Cr-1, n-2Cr-1, n-3Cr-1….r -1 Cr-1

But nCr = n − 1Cr − 1 + n − 1Cr.

Start with the toe : nCr=n − 1Cr − 1+ n − 1Cr.

Using the rule again, n − 1Cr=n− 2Cr − 1+n − 2Cr, and substituting into the stem equation

 nCr = n − 1Cr − 1+(n − 1Cr)=n − 1Cr − 1+(n − 2Cr − 1 + n − 2Cr).

Keep using the rule until you obtain

n Cr = n − 1Cr − 1+n − 2Cr − 1+ n − 3Cr − 1+ … + r + 1Cr − 1+ rCr− 1+rCr.

But = rCr = 1 = r − 1Cr − 1

So n Cr = n − 1Cr − 1+n − 2Cr − 1+ n − 3Cr − 1+ … +r + 1Cr − 1+rCr− 1+ r − 1C r − 1.

Therefore the sum of the cells in the stem is equal to the cell that represents the toe.

A delightful site supplied by Dr Dennis Ireland MLC is

<https://theconversation.com/the-12-days-of-pascals-triangular-christmas-21479>

**MERSENNE PRIMES**

## A Mersenne primeis a prime number of the form Mn = 2n – 1.

## Mersenne primes are named after the French monk Marin Mersenne who studied them in the early 17th century.

The first four Mersenne numbers are 1, 3, 7, 15.

(a) Determine a way to find these numbers on the Pascal Triangle below:

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(b) Hence determine the next three Mersenne prime numbers.

(c) Find an expression in terms of nCr for the fifth Mersenne number.

(d) Explain why 255 is a Mersenne prime.

(e) (i) Find a prime number between 1000 and 2000

 (ii) Explain a method to find the prime number you found in (i) using Pascal’s triangle.

(f) Prove that any Mersenne prime Mn may be expressed by 1 + 2 + 22 + 23 + .... + 2n-1

(g) Prove that is prime.

**ANSWER**

(a) To find the nth Mersenne prime number, add up all the terms of Pascal’s triangle for the first n

 rows.

(b) 31, 63, 127

(c) M5 = oCo + 1Co + 1C1 + 2Co + 2C1 + 2C2 + 3Co + 3C1 + 3C2 + 3C3

 + 4Co + 4C1 + 4C2 + 4C3 + 4C4 + 5Co + 5C1 + 5C2 + 5C3 + 5C4 + 5C5

(d) 255 = 256 – 1 = 28 -1

 ∴ 255 = M6, the 6th Mersenne prime.

(e) (i) 210 = 1024

 ∴ 1023 = M10

 (ii) To find the 10th Mersenne prime number, add up all the terms of Pascal’s triangle for the

 first 10 rows.

(f) Method 1

 1 + 2 + 22 + 23 + .... + 2n-1 is the sum of n terms of a geometric progression.

 Therefore 1 + 2 + 22 + 23 + .... + 2n-1 =  where a = 1, r = 2 and n – n

 1 + 2 + 22 + 23 + .... + 2n-1 =  which is the nth Mersenne prime.

 Method 2

 Mn is equal to the sum of the first n lines of Pascal’s triangle.

 Mn = (oCo ) + (1Co + 1C1) + ( 2Co + 2C1 + 2C2) + ...+ ( n-1Co + n-1C1 + -1C2 +... n-1Cn-1)

 = 20 + 21 +22 +... + 23 + .... + 2n-1

 = 1 + 2 + 22 + 23 + .... + 2n-1

(g) =1 + 2 + 22 + 23 + .... + 2n-1 = Mn (from (f))

 which is a Mersenne prime so is a prime number.

**FINAL FUN**

Colour in black all the odd numbers in the Pascal’s triangle below:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |



Extend to 15 lines.

You will have produced a portion of the Sierpinski Triangle.

Look up fractals and Sierpinski Triangle on the internet.

*Kindly proofed and extra suggestions by Dr Dennis Ireland and his staff at MLC.*